A geometric approach to variance analysis in system identification

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Outline

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Intro: *The* Asymptotic Covariance Matrix

We all know that under certain conditions

$$\sqrt{N}\left(\hat{\theta}_N - \theta_o\right) \in \mathrm{As}\mathcal{N}\left(0, P\right)$$

where P also is the *asymptotic covariance matrix*

$$P = \mathrm{AsCov}\,\hat{\theta}_N \triangleq \lim_{N\to\infty} N \cdot \mathbf{E}\left[(\hat{\theta}_N - \mathbf{E}\hat{\theta}_N)^T \overline{(\hat{\theta}_N - \mathbf{E}\hat{\theta}_N)}\right]$$

(we assume vectors to be row vectors.)

Intro: Pandora's (Black) Box

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  - Model order
  - Open vs closed loop
  - Input channels and input excitation
  - Sensor channels
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Many have contributed to revealing the secrets of P!

Intro: This talk

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Yet another attempt to decipher P,

but through a backdoor:

- Geometrical interpretation

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where $\Psi : \mathbb{C} \to \mathbb{C}^{n \times m}$, for some integer $m > 0$ depending on the model structure,
and where

$$\langle \Psi, \Psi \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{j\omega}) \Psi^*(e^{j\omega}) \mathrm{d}\omega$$

(inner products between the rows of Ψ)

The structure of P: Example 1 - FIR models

$$y_t = \sum_{k=1}^{n} \theta_k \, u_{t-k} + e_t = \theta \varphi_t^T + e_t$$

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$$\mathbf{E} \left[\varphi_t^T \varphi_t \right] = \begin{bmatrix} r_0 & r_1 & \dots & r_{n-1} \\ r_1 & r_0 & \dots & r_{n-2} \\ \vdots & & & \vdots \\ r_{n-1} & r_{n-2} & \dots & r_0 \end{bmatrix}$$

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$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_n(e^{j\omega}) \Phi_u(\omega) \Gamma_n^*(e^{j\omega})$$

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$$\Rightarrow \; P = \langle \Gamma_n \Phi_u^{1/2} / \sqrt{\lambda_o}, \Gamma_n \Phi_u^{1/2} / \sqrt{\lambda_o} \rangle = \langle \Psi, \Psi \rangle$$

The structure of P: Example 2 - NFIR models

$$y_t = \sum_{k=1}^{n} \theta_k \, u_{t-k} + \sum_{k=n+1}^{m} \theta_k \, (u_{t-(k-n)})^2 + e_t$$

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$$\begin{aligned} \varphi_t &= [u_{t-1}, \ldots, u_{t-n}, (u_{t-1})^2, \ldots, (u_{t-m})^2] \\ &= (M(q) z_t)^T \end{aligned}$$

where

$$M(q) = \begin{bmatrix} \Gamma_n(q) & 0 \\ 0 & \Gamma_m(q) \end{bmatrix}, \quad z_t = \begin{bmatrix} u_t \\ (u_t)^2 \end{bmatrix}$$

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$$\Rightarrow \quad \Psi = \frac{1}{\sqrt{\lambda_o}} M \Phi_z^{1/2}.$$

where $\Phi_z^{1/2}$ is a Cholesky factor of Φ_z, the spectrum of z.

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- impulse response
- system gain

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$$\text{AsCov } J(\hat{\theta}_N) = \Lambda^T \left[\langle \Psi, \Psi \rangle\right]^{-1} \overline{\Lambda}$$

where

$$\Lambda \triangleq J'(\theta_o) \in \mathbb{C}^{n\times q}$$

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Least squares estimation

$$Y = \theta\Phi + E$$

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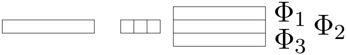

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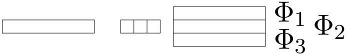

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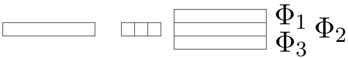

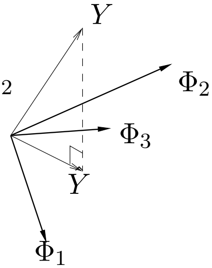

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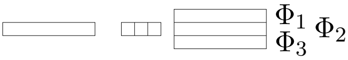

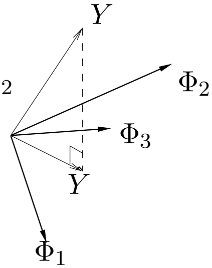

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Let the rows of $X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ span a (closed) subspace $\mathcal{S}_X$ to a Hilbert space $\mathcal{H}$.

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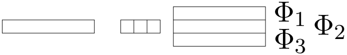

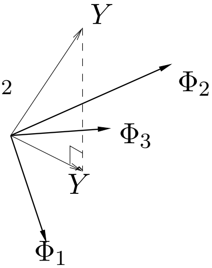

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Let the rows of $X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ span a (closed) subspace $\mathcal{S}_X$ to a Hilbert space $\mathcal{H}$.
Then the projection of $f \in \mathcal{H}$ on $\mathcal{S}_X$ is given by

$$\hat{f} = \langle f, X \rangle \left[\langle X, X \rangle\right]^{-1} X$$

A refresher on orthogonal projection: MV-case

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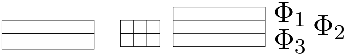

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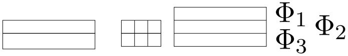

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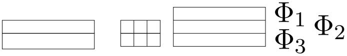

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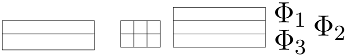

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$$\hat{f} = \langle f, X \rangle \left[\langle X, X \rangle\right]^{-1} X$$

where $f = \begin{bmatrix} f_1 & f_2 & \ldots & f_n \end{bmatrix}^T$

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The “norm” of the projection:

$$\langle \hat{f}, \hat{f} \rangle = \langle f, X \rangle \left[\langle X, X \rangle \right]^{-1} \langle X, f \rangle$$

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Take home message

Scalar quantities: The asymptotic variance is the squared norm of γ projected onto the subspace spanned by the rows of Ψ.

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A geometric interpretation of P

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Decoupling!

The main contribution in this talk: A methodology to analyze the asymptotic covariance matrix.

Examples will be used to illustrate this.

Outline

Example: FIR system

$$
\begin{aligned}
y_t &= \sum_{k=1}^{n} \theta_k \, u_{t-k} + e_t = \theta \varphi_t^T + e_t \\
P &= \left[\langle \Psi, \Psi \rangle\right]^{-1} \\
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Structural results: Adding parameters

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True order $n = n_o$, $J = J(\theta_1, \ldots, \theta_{n_o})$.

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True order $n = n_o$, $J = J(\theta_1, \ldots, \theta_{n_o})$.
What happens with the accuracy if we over-model ($n > n_o$)?

Adding parameters: FIR example

$$\text{AsCov } J(\hat{\theta}_N) = \begin{bmatrix} \Lambda \\ 0_{(n-n_o)\times q} \end{bmatrix}^* \left[\langle \Omega, \Omega \rangle\right]^{-1} \begin{bmatrix} \Lambda \\ 0_{(n-n_o)\times q} \end{bmatrix}$$

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where

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$$\Gamma_{m,n}(z) = \begin{bmatrix} z^{-(m+1)} & \dots & z^{-n} \end{bmatrix}^T.$$

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where

$$\Gamma_{m,n}(z) = \begin{bmatrix} z^{-(m+1)} & \dots & z^{-n} \end{bmatrix}^T .$$

The question is now how large $\Lambda^* [\langle \Psi, \Psi \rangle]^{-1} \Lambda$ is in comparison with the expression above.

Structural results: Adding parameters

Geometrical result

Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of $\mathcal{L}_2^m$ such that $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{L}_2^m$ and let $\gamma \in \mathcal{L}_2^{q \times m}$. It then holds that

$$\begin{aligned} &\langle \mathrm{Proj}_{\mathcal{Y}}\{\gamma\}, \mathrm{Proj}_{\mathcal{Y}}\{\gamma\} \rangle - \langle \mathrm{Proj}_{\mathcal{X}}\{\gamma\}, \mathrm{Proj}_{\mathcal{X}}\{\gamma\} \rangle \\ &= \langle \mathrm{Proj}_{\mathcal{X}^\perp(\mathcal{Y})}\{\gamma\}, \mathrm{Proj}_{\mathcal{X}^\perp(\mathcal{Y})}\{\gamma\} \rangle \geq 0 \end{aligned}$$

($\mathcal{X}^\perp(\mathcal{Y})$ denotes the orthogonal complement of $\mathcal{X}$ in $\mathcal{Y}$)

Structural results: Adding parameters

Geometrical result

Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of $\mathcal{L}_2^m$ such that $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{L}_2^m$ and let $\gamma \in \mathcal{L}_2^{q \times m}$. It then holds that

$$\begin{aligned} &\langle \mathrm{Proj}_{\mathcal{Y}}\{\gamma\}, \mathrm{Proj}_{\mathcal{Y}}\{\gamma\} \rangle - \langle \mathrm{Proj}_{\mathcal{X}}\{\gamma\}, \mathrm{Proj}_{\mathcal{X}}\{\gamma\} \rangle \\ &= \langle \mathrm{Proj}_{\mathcal{X}^\perp(\mathcal{Y})}\{\gamma\}, \mathrm{Proj}_{\mathcal{X}^\perp(\mathcal{Y})}\{\gamma\} \rangle \geq 0 \end{aligned}$$

($\mathcal{X}^\perp(\mathcal{Y})$ denotes the orthogonal complement of $\mathcal{X}$ in $\mathcal{Y}$)

- The asymptotic covariance can never decrease when parameters are added

Structural results: Adding parameters

Geometrical result

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\end{aligned}$$

($\mathcal{X}^\perp(\mathcal{Y})$ denotes the orthogonal complement of $\mathcal{X}$ in $\mathcal{Y}$)

- The asymptotic covariance can never decrease when parameters are added
- Above expression provides a quantification for the increase

Structural results: Adding parameters

Geometrical result

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($\mathcal{X}^\perp(\mathcal{Y})$ denotes the orthogonal complement of $\mathcal{X}$ in $\mathcal{Y}$)

- The asymptotic covariance can never decrease when parameters are added
- Above expression provides a quantification for the increase
- No increase if γ orthogonal to the orthogonal complement of X in $\mathcal{Y}$

Adding parameters: FIR example

$$\text{AsCov } J(\hat{\theta}_N) = \begin{bmatrix} \Lambda \\ 0_{(n-n_o)\times q} \end{bmatrix}^* [\langle \Omega, \Omega \rangle]^{-1} \begin{bmatrix} \Lambda \\ 0_{(n-n_o)\times q} \end{bmatrix}$$

where $\Omega = \Gamma_n \frac{\Phi_u^{1/2}}{\sqrt{\lambda_o}} = \begin{bmatrix} \Gamma_{n_o} \\ \Gamma_{n_o,n} \end{bmatrix} \frac{\Phi_u^{1/2}}{\sqrt{\lambda_o}} \triangleq \begin{bmatrix} \Psi \\ \Phi \end{bmatrix}$

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- No increase:

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- Strict increase:

Adding parameters: FIR example

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- No increase: $\langle \Phi, \Psi \rangle = 0 \Leftrightarrow \mathcal{S}_\Phi \perp \mathcal{S}_\Psi \Rightarrow$ $\mathcal{S}_\Psi^\perp(\mathcal{S}_\Omega) = \mathcal{S}_\Phi \Rightarrow \text{Proj}_{\mathcal{S}_\Psi^\perp(\mathcal{S}_\Omega)}\{\gamma\} = 0$ ($\Leftarrow$ also holds)
- Strict increase: $\langle \Phi, \Psi \rangle$ full column rank.

Structural results: Adding parameters - FIR example

- No increase: If and only if $\langle \Phi, \Psi \rangle = 0$
- Strict increase: $\langle \Phi, \Psi \rangle$ full column rank.

Structural results: Adding parameters - FIR example

- No increase: If and only if $\langle \Phi, \Psi \rangle = 0$
- Strict increase: $\langle \Phi, \Psi \rangle$ full column rank.

FIR example:

$$\begin{aligned}
\langle \Phi, \Psi \rangle =& \langle \Gamma_{n_o,n} \Phi_u^{1/2}, \Gamma_{n_o} \Phi_u^{1/2} \rangle \\
&= R_{n_o,n} \triangleq \begin{bmatrix} r_{n_o} & r_{n_o-1} & \cdots & r_1 \\ r_{n_o+1} & r_{n_o} & \cdots & r_2 \\ \vdots & & & \vdots \\ r_{n-1} & r_{n-2} & \cdots & r_{n-n_o} \end{bmatrix},
\end{aligned}$$

Structural results: Adding parameters - FIR example

- No increase: If and only if $\langle \Phi, \Psi \rangle = 0$
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- No increase: If and only if $r_1 = \ldots = r_{n-1} = 0$
- Strict increase: If and only if $R_{n_o,n}$ has full column rank ($n \geq 2n_o$ necessary condition)

Structural results: Adding parameters

- No increase: If and only if $\langle \Phi, \Psi \rangle = 0$
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Derivation not tied to the FIR example.

Structural results: Adding parameters

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Derivation not tied to the FIR example.

Generalization immediate!

Structural results: Adding parameters - NFIR

$$y_t = \sum_{k=1}^{n} \theta_k \, u_{t-k} + \sum_{k=n+1}^{m} \theta_k \, (u_{t-(k-n)})^2 + e_t = \theta \varphi_t^T + e_t$$

$$\Psi = \frac{1}{\sqrt{\lambda_o}} M \Phi_z^{1/2}.$$

where $\Phi_z^{1/2}$ is a Cholesky factor of Φ_z, the spectrum of $z = [u_t \; u_t^2]^T$.

Structural results: Adding parameters - NFIR

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White input with $\mathbf{E}[u_t^3] \neq 0$:

Structural results: Adding parameters - NFIR

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White input with $\mathbf{E}[u_t^3] \neq 0$:

$$\Psi \triangleq \begin{bmatrix} \sqrt{\mathbf{E}[u_t^2]} \; \Gamma_n & 0 \\ \frac{\mathbf{E}[u_t^3]}{\sqrt{\mathbf{E}[u_t^2]}} \; \Gamma_n & \sqrt{\mathbf{E}[u_t^4] - \frac{\mathbf{E}^2[u_t^3]}{\mathbf{E}[u_t^2]}} \; \Gamma_n \end{bmatrix}$$

Structural results: Adding parameters - NFIR

$$y_t = \sum_{k=1}^{n} \theta_k \; u_{t-k} + \sum_{k=n+1}^{m} \theta_k \; (u_{t-(k-n)})^2 + e_t = \theta \varphi_t^T + e_t$$

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Compare the asymptotic covariance of the first $2n$ parameters when estimating $m = 2n$ and $m > 2n$ non-linear parameters.

Structural results: Adding parameters - NFIR

$$\Psi \triangleq \begin{bmatrix} \sqrt{\mathbf{E}[u_t^2]}\,\Gamma_n & 0 \\ \frac{\mathbf{E}[u_t^3]}{\sqrt{\mathbf{E}[u_t^2]}}\,\Gamma_n & \sqrt{\mathbf{E}[u_t^4] - \frac{\mathbf{E}^2[u_t^3]}{\mathbf{E}[u_t^2]}}\,\Gamma_n \end{bmatrix}$$

Structural results: Adding parameters - NFIR

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$$\Phi \triangleq \begin{bmatrix} \frac{\mathbf{E}[u_t^3]}{\sqrt{\mathbf{E}[u_t^2]}}\;\Gamma_{n,m} & \sqrt{\mathbf{E}[u_t^4] - \frac{\mathbf{E}^2[u_t^3]}{\mathbf{E}[u_t^2]}}\;\Gamma_{n,m} \end{bmatrix}$$

Structural results: Adding parameters - NFIR

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- $\mathcal{S}_\Psi \perp \mathcal{S}_\Phi \Rightarrow$ No increase in the asymptotic covariance of the linear parameters and the first n non-linear parameters from additional non-linear parameters.

Structural results: Adding parameters - NFIR

$$\Psi \triangleq \begin{bmatrix} \sqrt{\mathbf{E}[u_t^2]}\,\Gamma_n & 0 \\ \frac{\mathbf{E}[u_t^3]}{\sqrt{\mathbf{E}[u_t^2]}}\,\Gamma_n & \sqrt{\mathbf{E}[u_t^4] - \frac{\mathbf{E}^2[u_t^3]}{\mathbf{E}[u_t^2]}}\,\Gamma_n \end{bmatrix}$$

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- $\mathcal{S}_\Psi \perp \mathcal{S}_\Phi \Rightarrow$ No increase in the asymptotic covariance of the linear parameters and the first n non-linear parameters from additional non-linear parameters.
- Can also easily show that increase is strict for the linear parameters when more than n non-linear parameters are estimated.

Structural results: Adding sources

Structural results: Adding sources

FIR system with second input:

$$y_t = \sum_{k=1}^{n} \theta_k \left(u_{t-k} + w_{t-k}\right) + \sum_{k=1}^{m} \theta_{n+k}\, w_{t-k} + e_t$$

Structural results: Adding sources

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$$y_t = \sum_{k=1}^{n} \theta_k \left(u_{t-k} + w_{t-k}\right) + \sum_{k=1}^{m} \theta_{n+k}\, w_{t-k} + e_t$$

This system can be written in linear regression form using

$$\varphi_t^T = \begin{bmatrix} \Gamma_n & \Gamma_n \\ 0_{m\times 1} & \Gamma_m \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix}$$

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$$\Psi = \Psi_e \triangleq \begin{bmatrix} \Gamma_n & \Gamma_n \\ 0_{m\times 1} & \Gamma_m \end{bmatrix} \begin{bmatrix} \frac{\Phi_u^{1/2}}{\sqrt{\lambda_o}} & 0 \\ 0 & \frac{\Phi_w^{1/2}}{\sqrt{\lambda_o}} \end{bmatrix} = \begin{bmatrix} \frac{\Phi_u^{1/2}}{\sqrt{\lambda_o}} \Gamma_n & \frac{\Phi_w^{1/2}}{\sqrt{\lambda_o}} \Gamma_n \\ 0_{m\times 1} & \frac{\Phi_w^{1/2}}{\sqrt{\lambda_o}} \Gamma_m \end{bmatrix}$$

Structural results: Adding sources

FIR system with second input:

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Second column of Ψ due to the new input w, but also m new rows due to the additional parameters $\theta_{n+1}, \ldots, \theta_{n+m}$ that need to be estimated.

Structural results: Adding sources

FIR system with second input:

$$y_t = \sum_{k=1}^{n} \theta_k \left(u_{t-k} + w_{t-k}\right) + \sum_{k=1}^{m} \theta_{n+k}\, w_{t-k} + e_t$$

$$\Psi_e = \begin{bmatrix} \frac{\Phi_u^{1/2}}{\sqrt{\lambda_o}}\, \Gamma_n & \frac{\Phi_w^{1/2}}{\sqrt{\lambda_o}}\, \Gamma_n \\ 0_{m\times 1} & \frac{\Phi_w^{1/2}}{\sqrt{\lambda_o}}\, \Gamma_m \end{bmatrix}$$

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We're only interested in $\theta_1, \ldots, \theta_n$.

Structural results: Adding sources

FIR system with second input:

$$y_t = \sum_{k=1}^{n} \theta_k \left(u_{t-k} + w_{t-k}\right) + \sum_{k=1}^{m} \theta_{n+k}\, w_{t-k} + e_t$$

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We're only interested in $\theta_1, \ldots, \theta_n$.
Will using w increase the accuracy?

+ SNR in first terms increase
- m additional parameters has to be estimated

Structural results: Geometric analysis

Let $\Psi \in \mathcal{L}_2^{n\times m}$, let $\Lambda \in \mathbb{C}^{n\times q}$ and

$$\Psi_e = \begin{bmatrix} \Psi & \psi \\ 0_{\delta n\times m} & \eta \end{bmatrix}$$

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Assume that $\langle \Psi, \Psi \rangle > 0$ and $\langle \Psi_e, \Psi_e \rangle > 0$ and define

$$X \triangleq \Lambda^* \left[\langle \Psi, \Psi \rangle\right]^{-1} \Lambda - \begin{bmatrix} \Lambda \\ 0_{\delta n\times q} \end{bmatrix}^* \left[\langle \Psi_e, \Psi_e \rangle\right]^{-1} \begin{bmatrix} \Lambda \\ 0_{\delta n\times q} \end{bmatrix}$$

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Theorem

$$X = \langle \mathrm{Proj}_{\mathcal{S}^{\perp}_{\Psi_e}(\mathcal{S}_\Omega)}\{\gamma\}, \mathrm{Proj}_{\mathcal{S}^{\perp}_{\Psi_e}(\mathcal{S}_\Omega)}\{\gamma\} \rangle \geq 0$$

where $\gamma \triangleq \Lambda^* \left[\langle \Phi, \Phi \rangle\right]^{-1} \Phi$, $\Phi \triangleq \begin{bmatrix} \Psi & 0_{n\times \delta m} \end{bmatrix}$, $\Omega \triangleq \begin{bmatrix} \Psi_e \\ \Phi \end{bmatrix}$

Structural results: Adding sources

$$\Psi_e = \begin{bmatrix} \frac{\Phi_u^{1/2}}{\sqrt{\lambda_o}}\ \Gamma_n & \frac{\Phi_w^{1/2}}{\sqrt{\lambda_o}}\ \Gamma_n \\ 0_{m\times 1} & \frac{\Phi_w^{1/2}}{\sqrt{\lambda_o}}\ \Gamma_m \end{bmatrix}$$

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No improvement (regardless of the power of w).

Structural results: Adding sensors

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OE system (e_t is WGN with variance λ_e)

$$y_t = G(q, \theta)u_t + e_t$$

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$$z_t = M^o(q)G^o(q)u_t + w_t$$

where $\{w_t\}$ is WGN with variance $\lambda_w << \lambda_e$, increase the accuracy of the estimate of θ?

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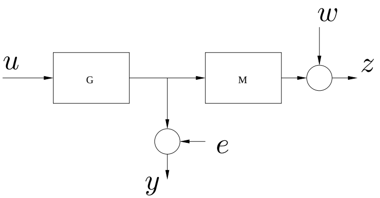

Structural results: Adding sensors

Sensor dynamics unknown and needs to be estimated:

$$M(q, \beta), \qquad \text{where } \beta \in \mathbb{R}^{1 \times n_\beta}.$$

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ML estimation with two sensors $\Rightarrow$

$$\Psi = \Psi_2 = \begin{bmatrix} \frac{R}{\sqrt{\lambda_e}} G'(\theta^o) & \frac{R}{\sqrt{\lambda_w}} M^o G'(\theta^o) \\ 0_{n_\beta \times 1} & \frac{R}{\sqrt{\lambda_w}} G^o M'(\beta^o) \end{bmatrix}$$

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No improvement whatsoever if $\frac{1}{\sqrt{\lambda_w}} M^o G'(\theta^o)$ spanned by rows of $\frac{1}{\sqrt{\lambda_w}} G^o M'(\beta^o)$.

Structural results: Adding sensors

Example: $\mathcal{L}_2$-gain estimation

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System G and sensor M of FIR type.

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$J(\theta) = \|\theta\|$

Structural results: Adding sensors

Example: $\mathcal{L}_2$-gain estimation

System G and sensor M of FIR type.

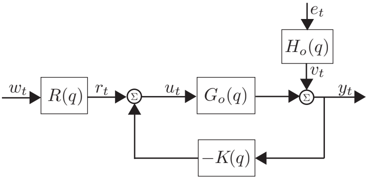
$$J(\theta) = \|\theta\|$$

There is no sensor that can improve the accuracy of J, even if the sensor only contains one single unknown parameter and the sensor is virtually noise free.

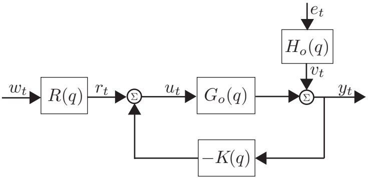
Outline

Properties of SISO LTI systems

Properties of SISO LTI systems



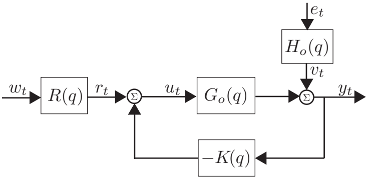
Properties of SISO LTI systems



$$y(t) = \begin{bmatrix} G(q,\theta) & H(q,\theta) \end{bmatrix} \begin{bmatrix} u(t) \\ e(t) \end{bmatrix} = T(q,\theta)\chi(t)$$

$$\chi(t) = \begin{bmatrix} S(q)R(q) & -C(q)S(q)H(q) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r(t) \\ e(t) \end{bmatrix} = U(q)\xi(t).$$

Properties of SISO LTI systems



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Closed loop: $y(t) = T(q,\theta)U(q)\xi(t)$

Prediction error gradient: $\Psi(z) = \frac{1}{\sqrt{\lambda_o}H_o(z)} T'(z,\theta^o)U_o(z)\Phi_\xi^{1/2}$

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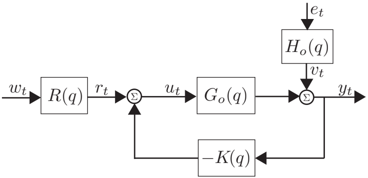

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Open loop: $\Psi(z) = \begin{bmatrix} \frac{R(z)}{\sqrt{\lambda_o}{}_o(z)} G'(z,\theta^o) & \frac{1}{H_o(z)} H'(z,\theta^o) \end{bmatrix}$

Properties of SISO linear systems

Which quantities are difficult to estimate?

- Frequency functions
- Impulse response coefficients
- System gains (e.g. $\mathcal{L}_2$)
- Zeros and poles
- Control design performance
-

Rules of thumbs??

Properties of SISO linear systems

Basic idea:

- Parametrize quantity of interest J in terms of the system impulse response (could also be done using the frequency function)
- Use re-parametrization together with geometrical approach

Properties of SISO linear systems

Theorem

The asymptotic variance of an estimated quantity $J(G(\cdot))$ is given by

$$\operatorname{AsVar} \hat{J} = \lambda_0 \left\| \Pr_{\mathcal{S}_\Psi} \left(\nabla J(z) \begin{bmatrix} \frac{H(z^{-1})}{S(z^{-1})R(z^{-1})} & 0 \end{bmatrix} \right) \right\|^2$$

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Is this really better???

The variance expression 1(3)

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Definition

$$\nabla J(z) = \sum_{k=1}^{\infty} \overline{\frac{\mathrm{d}J(g)}{\mathrm{d}g_k}} z^{-k}$$

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Examples: $J(\cdot)$ is

- the $\mathcal{L}_2$-gain $\Rightarrow \nabla J(z) = 2G(z)$
- a real nmp-zero $z_k \Rightarrow \nabla J(z) \propto \frac{1}{z - z_k^{-1}}$
- an impulse response coefficient $g_k \Rightarrow \nabla J(z) = z^{-k}$

The variance expression 2(3)

$$\operatorname{AsVar} \hat{J} = \lambda_0 \left\| \operatorname{Pr}_{\mathcal{S}_\Psi} \left(\nabla J(z) \begin{bmatrix} \frac{H(z^{-1})}{S(z^{-1})R(z^{-1})} & 0 \end{bmatrix} \right) \right\|^2$$

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$\frac{H(z)}{S(z)R(z)}$ is the spectral factor of $\frac{\Phi_v(z)}{\Phi_u^r(z)}$

The variance expression 3(3)

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The space $\mathcal{S}_\Psi$ is defined as the linear span of the rows of the predictor gradient Ψ. The space can be written as

$$\mathcal{S}_\Psi = \mathrm{span}\{H^{-1}(z)T'(z)U(z)\}$$

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Example:

For an FIR model structure in open loop the space is

$$\mathcal{S}_\Psi = \mathrm{span}\left\{ \begin{bmatrix} R(z)z^{-1} & 0 \end{bmatrix}, \cdots, \begin{bmatrix} R(z)z^{-n} & 0 \end{bmatrix} \right\}$$

where $\Phi_u(z) = R(z)R(z^{-1})$.

Properties of SISO LTI systems

$$\operatorname{AsVar}\hat{J} = \lambda_0 \left\| \Pr_{\mathcal{S}_\Psi} \left(\nabla J(z) \begin{bmatrix} \frac{H(z^{-1})}{S(z^{-1})R(z^{-1})} & 0 \end{bmatrix} \right) \right\|^2$$

What does the expression tell us?

Properties of SISO LTI systems

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How the asymptotic variance depends on

- Input signal
- Feedback control
- Model structure
- The property itself $J(\cdot)$

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Influence decoupled!

Properties of SISO LTI systems: Upper bounds

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Still think it's too messy?

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Then forget about the projection:

$$\operatorname{AsVar} \hat{J} \leq \lambda_0 \left\| \left(\nabla J(z) \begin{bmatrix} \frac{H(z^{-1})}{S(z^{-1})R(z^{-1})} & 0 \end{bmatrix} \right) \right\|^2$$

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$$\operatorname{AsVar} \hat{J} \leq \|\nabla J\|^2_{\Phi_v/\Phi_u^r}$$

Independent of the model structure!

Properties of SISO LTI systems: Examples

$$\operatorname{AsVar} \hat{J} \leq \|\nabla J\|^2_{\Phi_v/\Phi_u^r}$$

Properties of SISO LTI systems: Examples

$$\mathrm{AsVar}\, \hat{J} \leq \|\nabla J\|^2_{\Phi_v/\Phi_u^r}$$

Quantity	Upper bound
impulse response coef.	$\|1\|^2_{\frac{\Phi_v}{\Phi_u^r}}$
L_2-gain	$\frac{4}{N}\|G\|^2_{\frac{\Phi_v}{\Phi_u^r}}$
real nmp-zero	$\frac{1}{\lvert\widetilde{G}_o(z_o)\rvert^2}\left\|\frac{1}{1-z_o^{-1}z^{-1}}\right\|^2_{\frac{\Phi_v}{\Phi_u^r}}$

$$(\widetilde{G}_o(z) = G_o(z)/(1 - z_o z^{-1}))$$

Properties of LTI systems: Frequency function

For an estimate of the frequency function $\sum_{k=1}^{\infty} g_k e^{-j\omega k}$ we have

$$\frac{dJ}{dg_k} = e^{-j\omega k}$$

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and the expression reduces to

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Same expression as given by Xie& Ljung (2001), Ninness&Hjalmarsson(2003,2004)

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Minimum phase system with 1 time delay $\Rightarrow$ MV-controller

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What can we say about

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Minimum cost: $\delta J_{ho} = \lambda_o m$

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Write $\mathcal{B}_k = [\mathcal{B}_k^G,\ \mathcal{B}_k^H]$. Then simple algebra gives

$$\delta J = \lambda_o\sum_{k=1}^{n}\left\|Z_o\,\mathcal{B}_k^G\right\|^2 + \lambda_o\sum_{k=1}^{n}\left\|\mathcal{B}_k^H\right\|^2$$

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Result ii) is a generalization of Theorem 1 in Hildebrand&Solari 2007 which covers the case B_o constant and $n_c \geq n_a$.

Is open loop or closed loop identification better when the input spectrum due to the reference signal is fix?

Properties of SISO LTI systems: Open vs closed

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$\Rightarrow$ Open or closed loop operation only affects the subspace $\mathcal{S}_\Psi$

Box Jenkins models

$$\mathcal{S}_{\Psi_{ol}} = \mathrm{Span}\left\{ \begin{bmatrix}\alpha_1\\0\end{bmatrix}^T, \cdots, \begin{bmatrix}\alpha_{n_\theta}\\0\end{bmatrix}^T, \begin{bmatrix}0\\\gamma_1\end{bmatrix}^T, \cdots, \begin{bmatrix}0\\\gamma_{n_\eta}\end{bmatrix}^T \right\}$$

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$$\text{AsVar}\, J(\hat{\theta}_{cl}) \leq \text{AsVar}\, J(\hat{\theta}_{ol})$$

c.f. Bombois et al and Aguero et al.

Outline

1. Introduction
2. The structure of the asymptotic covariance matrix P
3. A refresher on orthogonal projection
4. A geometric interpretation of P
5. Structural results
6. Analysis of properties of SISO LTI systems
7. **A non-linear sample**
8. Input design
9. Summary

A nonlinear sample: Hammerstein systems

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Generalization of Ninness and Gibson 2002.

Outline

Open loop input design

$$\min_{\Phi_u} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Phi_u(e^{j\omega})}{\Phi_v(e^{j\omega})} \, d\omega$$

$$\text{s.t. } \operatorname{AsVar} J(\hat{\theta}_N) \leq \frac{1}{\alpha}$$

(α is the precision)

Open loop input design: FIR example (again)

$$\begin{aligned}\operatorname{AsVar}\hat{J} &= \lambda_0 \left\| \Pr_{\mathcal{S}_\Psi} \left(\nabla J(z) \begin{bmatrix} \frac{H(z^{-1})}{S(z^{-1})R(z^{-1})} & 0 \end{bmatrix} \right) \right\|^2 \\ &= \lambda_0 \left\| \Pr_{\mathcal{S}_\Psi} \left(\nabla J(z) \begin{bmatrix} \frac{1}{R(z^{-1})} & 0 \end{bmatrix} \right) \right\|^2\end{aligned}$$

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$$\mathcal{S}_\Psi = \operatorname{span}\left\{\begin{bmatrix}R(z)z^{-1} & 0\end{bmatrix}, \cdots, \begin{bmatrix}R(z)z^{-n} & 0\end{bmatrix}\right\}$$

where $\Phi_u(z) = R(z)R(z^{-1})$.

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- Generalization to identification of a frequency band in poster session on Monday (Rojas, Barenthin and Hjalmarsson)

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Summary

Main contribution

The methodology!